7. Chernous'ko, F. L., Some problems of optimal control with a small parameter. PMM Vol. 32, № 1, 1968.
8. Kolmanovskii, V. B. and Chernous'ko, F. L., Optimal control problems with incomplete information. Proc. Fourth Winter School on Mathematical Programing and Related Questions, № 1, Moscow, 1971.
9. Solianik, A. I. and Chernous'ko, F. L., Approximate synthesis method for otimal control of a system subjected to random perturbations. PMM Vol. 36, № 5, 1972.
10. Bratus', A. S., Approximate solution of the Bellman's equation for a class of optimal terminal state control problems. PMM Vol. 37, № 3, 1973.
11. Bratus', A. S., Method for the approximate solution of the Bellman equation for problems of optimal control of systems subject to random perturbations. PMM Vol. 39, № 2, 1975.

Translated by N. H. C.

UDC 532.546.013.2

THE NONLINEAR PROBLEM OF UNSTEADY FILTRATION OF HEAVY FLUID WITH A FREE SURFACE

PMM Vol. 39, № 6, 1975, pp. 1017-1022
L. A. GALIN
(Moscow)
(Received August 6, 1975)

A solution is derived for the problem of unsteady motion of heavy fluid with a free surface in the vertical plane in a porous medium. Such problems are encountered in irrigation and land improvement schemes in connection with the filtration of ground waters. To use numerical and approximate methods for obtaining a solution of this fairly difficult problem one must be sure of its existence. The case when the heavy fluid occupies, at the initial instant of time, a finite region, is considered. An earlier investigation of this problem by the author [1] was based on some other assumptions with the heavy fluid occupying a semi-infinite region.

Let region L occupied by a heavy fluid be mapped onto a unit circle in plane ζ by means of function $z(\zeta, t)$, where the time t is a parameter. At the initial instant of time

$$z(\zeta, 0) = z_0(\zeta) \tag{1}$$

In this representation the coordinate origin in the ζ-plane corresponds to a drain in the L region. Function $z(\zeta, t)$ which depends on the complex variable ζ and on time t must satisfy some boundary condition at subsequent instants.

The velocity potential of the motion of a heavy fluid is

$$\varphi = -k\left(\frac{p}{\rho g} + y\right) \tag{2}$$

where p is the pressure, k is the filtration coefficient, ρ is the density, and g is the acceleration of gravity. The velocity components are

$$v_x = \frac{\partial \varphi}{\partial x}, \quad v_y = \frac{\partial \varphi}{\partial y}$$

For determining the rate of boundary translation these quantities must be divided by the porosity coefficient m.

We assume that along the contour of region L the pressure is p_1, i.e.

$$p_L = p_1 \tag{3}$$

This occurs when a heavy fluid is introduced into the porous medium region which initially did not contain any fluid. In the absence of a drain the region occupied by the fluid moves downward at a constant velocity. This follows from the assumption of constant pressure p_1 throughout the region.

Hence it follows from (2) that

$$v_x = 0, \quad v_y = k$$

We assume that at the contour of the circular drain bore the pressure is $p_2 < p_1$. Consequently

$$p \mid_{z=\delta} = p_2 \tag{4}$$

where δ is the radius of the bore.

The boundary condition along the contour of region L, which must be satisfied by function $z(\zeta, t)$, was established by the author in [2]. It is derived here in greater detail than it was done in [2].

From (2) we obtain for the velocity potential

$$\varphi = -\frac{k}{\rho g} p - ky$$

hence

$$p = -\frac{\rho g}{k} \varphi - \rho g y$$

We introduce into the analysis the variable ζ. Since in the unit circle the region ζ the pressure along the external $(|\zeta| = 1)$ and the internal $(|\zeta| = \delta_1)$ contours is constant, p is defined by a logarithmic function of ζ; δ_1 is the drain radius in the ζ-plane. It follows from this that at the unit circle contour

$$\frac{\partial p}{\partial n} = -\frac{pg}{k}\frac{\partial y}{\partial n} - \rho g \frac{\partial y}{\partial n} = A \left| \frac{\partial z}{\partial \zeta} \right|^{-1}_{\zeta = e^{i\vartheta}}, \quad A = \frac{p_2 - p_1}{\lg \delta_1} \tag{5}$$

which implies that

$$\frac{\partial \varphi}{\partial n} = -k\frac{\partial y}{\partial n} - \frac{kA}{\rho g}\left| \frac{\partial z}{\partial \zeta} \right|^{-1}_{\zeta = e^{i\vartheta}} \tag{6}$$

Let us now determine $\partial y / \partial n$. We have

$$\frac{\partial y}{\partial n} = \sin\left[\arg\left(\zeta \frac{\partial z}{\partial \zeta}\right)_{\zeta = e^{i\vartheta}}\right] = \operatorname{Im}\left[\zeta \frac{\partial z}{\partial \zeta}\right]_{\zeta = e^{i\vartheta}} \cdot \left| \frac{\partial z}{\partial \zeta} \right|^{-1}_{\zeta = e^{i\vartheta}}$$

Since

$$\operatorname{Im}\left[\zeta \frac{\partial z}{\partial \zeta}\right] = \operatorname{Re}\left[i\zeta \frac{\partial z}{\partial \zeta}\right]$$

we finally obtain

$$\frac{\partial y}{\partial n} = \operatorname{Re}\left[-\zeta \frac{\partial z}{\partial \zeta}\right]_{\zeta = e^{i\vartheta}} \cdot \left| \frac{\partial z}{\partial \zeta} \right|^{-1}_{\zeta = e^{i\vartheta}} \tag{7}$$

Substituting (7) into (8) (*) after transformation, for the velocity of the fluid motion along the normal to the contour of L at the corresponding points of the unit circle we obtain the formula

*) Translator's note: Evidently an error in the Russian original.

$$\frac{\partial y}{\partial n} = -k\left|\frac{\partial z}{\partial \zeta}\right|^{-1}_{\zeta=e^{i\theta}} \cdot \left[\frac{A}{\rho g} - \operatorname{Re}\left(i\zeta\frac{\partial z}{\partial \zeta}\right)_{\zeta=e^{i\theta}}\right] \tag{8}$$

The translation of a point along the contour of region L during time t is

$$\varepsilon_1 = \frac{1}{m}\frac{\partial \varphi}{\partial n}dt \tag{9}$$

where m is the coefficient of the medium porosity. In such case the translation along the normal to the contour of region L and, also, at points of the unit circle contour is

$$\varepsilon = -\frac{k}{m}\left|\frac{\partial z}{\partial \zeta}\right|^{-2}_{\zeta=e^{i\theta}} \cdot \left[\frac{p_2 - p_1}{\rho g \lg \delta_1} - \operatorname{Re}\left(i\zeta\frac{\partial z}{\partial \zeta}\right)_{\zeta=e^{i\theta}}\right]dt \tag{10}$$

where the formulas for A and (8) have been used. During the time interval dt we obtain in the region of variable ζ a contour which is close to the unit circle.

The function that maps in this case a nearly circular contour onto the circle is of the form

$$\zeta_1(\zeta) = \zeta - \zeta \underset{\gamma}{S}\left\{\left[-\frac{k(p_2-p_1)}{\rho g \lg \delta_1} + \frac{k}{m}\operatorname{Re}\left(i\zeta\frac{\partial z}{\partial \zeta}\right)_{\zeta=e^{i\theta}}\right]dt\right\} \tag{11}$$

where the Schwarz symbol $\underset{\gamma}{S}$, is used for brevity to denote the analytic function whose value at the unit circle contour appears within braces. We may write

$$z(\zeta, t+dt) = z\left[\zeta - \zeta \underset{\gamma}{S}\left\{\left[-\frac{k(p_2-p_1)}{\rho g \lg \delta_1} + \frac{k}{m}\left(i\zeta\frac{\partial z}{\partial \zeta}\right)_{\zeta=e^{i\theta}}\right]dt\right\}\right] \tag{12}$$

from which

$$\operatorname{Re}\left\{\left(\frac{\partial \bar z}{\partial t}\frac{\partial z}{\partial \zeta}\right)\frac{1}{\zeta}\right\}_{\zeta=e^{i\theta}} = \left[-\frac{k(p_2-p_1)}{\rho g\, m \lg \delta_1} + \frac{k}{m}\operatorname{Re}\left(i\zeta\frac{\partial z}{\partial \zeta}\right)_{\zeta=e^{i\theta}}\right]\left|\frac{dz}{d\zeta}\right|^{-2}_{\zeta=e^{i\theta}} \tag{13}$$

After transformation of this formula, we finally obtain

$$\operatorname{Re}\left\{\frac{\partial \bar z}{\partial t}\zeta\frac{\partial z}{\partial \zeta} - \frac{k}{m}i\zeta\frac{\partial z}{\partial \zeta}\right\}_{\zeta=e^{i\theta}} = -\frac{k}{\rho g m}\frac{p_2 - p_1}{(\lg \delta - \lg|\partial z/\partial \zeta|_{\zeta=0})} \tag{14}$$

where it is taken into account that $\lg \delta_1 = \lg \delta - \lg |\partial z / \partial \zeta|_{\zeta=0}$. This condition was obtained in [3] in another way.

Below we use the notation

$$\frac{k}{m} = \lambda, \qquad \frac{k(p_2-p_1)}{\rho g m} = \mu$$

and assume that at the initial instant of time the region occupied by the heavy fluid is mapped onto the unit circle by function

$$z(\zeta, 0) = z_0(\zeta) = a_1^*\zeta + a_2^*\zeta^2 + a_3^*\zeta^3 + \ldots \tag{15}$$

The number of terms in the right-hand part can be finite. When it is so, this function is at the initial instant of time a polynomial of finite order. For subsequent instants of time the function is sought in the form

$$z(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2 + a_3(t)\zeta^3 + \ldots \tag{16}$$

In that case functions $a_1(t)$, $a_2(t)$ . . . and their initial values are real, and the region at the initial instant of time is symmetric and retains its symmetry at all subsequent instants. We then evidently have the condition

$$\left|\frac{\partial z}{\partial \zeta}\right|_{\zeta=0} = a_1(t)$$

Thus the right-hand part of condition (14) is

$$\mu/(\lg \delta - \lg a_1(t))$$

Owing to this condition, equality (14) assumes the form

$$\operatorname{Re}\left[\frac{\partial \bar{z}}{\partial t}\zeta\frac{\partial z}{\partial \zeta} - i\lambda\zeta\frac{\partial z}{\partial \zeta}\right]_{\zeta=e^{i\vartheta}} = \frac{\mu}{\lg \delta - \lg a_1(t)} \tag{17}$$

Let us now substitute series (16) into formula (17) and consider the case of a finite number of terms, for instance, four. Below we use functions of time without the argument. We obtain

$$\operatorname{Re}\left[\frac{\partial \bar{z}}{\partial t}\zeta\frac{\partial z}{\partial \zeta} - i\lambda\zeta\frac{\partial z}{\partial \zeta}\right]_{\zeta=e^{i\vartheta}} = \frac{\mu}{\lg \delta - \lg a_1(t)} = \operatorname{Re}\left\{\left[a_1'\frac{1}{\zeta} + a_2'\frac{1}{\zeta^2} + \right.\right. \tag{18}$$
$$\left. a_3'\frac{1}{\zeta^3} + a_4'\frac{1}{\zeta^4}\right][a_1\zeta + 2a_2\zeta^2 + 3a_3\zeta^3 + 4a_4\zeta^4] + \lambda a_1\zeta +$$
$$\left. 2\lambda a_2\zeta^2 + 3\lambda a_3\zeta^3 + 4\lambda a_4\zeta^4\right\}_{\zeta=e^{i\vartheta}}$$

Carrying out in (18) the multiplication, we obtain a series which contains the powers ζ^{-3}, ζ^{-2}, ζ^{-1}, ζ^0, ζ^1, ζ^2, ζ^3 and ζ^4 multiplied by some coefficients containing certain functions a_n and a_m'.

Noting that

$$\operatorname{Re}[\zeta^n]_{\zeta=e^{i\vartheta}} = \operatorname{Re}[\zeta^{-n}]_{\zeta=e^{i\vartheta}} = \cos n\nu$$

from (18) we obtain the following formula:

$$\operatorname{Re}\left[\frac{\partial \bar{z}}{\partial \zeta}\zeta\frac{\partial z}{\partial \zeta} - i\lambda\zeta\frac{\partial z}{\partial \zeta}\right]_{\zeta=e^{i\vartheta}} - \frac{\mu}{\lg \delta_1 - \lg a} = [a_1'a_4 + 4a_1a_4' + 3\lambda a_3] \times \tag{19}$$
$$\cos 3\nu + [a_1a_3' + 2a_2a_4' + 3a_1'a_3 + 4a_2'a_4 + 2\lambda a_2]\cos 2\nu +$$
$$[a_1a_2' + 2a_2a_3' + 3a_3a_4' + 2a_1'a_2 + 3a_2'a_3 + 4a_3'a_4 + \lambda a_1] \times$$
$$\cos\nu + [a_1a_1' + 2a_3a_2' + 3a_3a_3' + 4a_4a_4'] + 4a_4\cos 4\nu$$

We equate the coefficients at $\cos 3\nu$, $\cos 2\nu$ and $\cos\nu$ to zero, and obtain the following system of four equations and initial conditions for four functions:

$$a_1a_1' + 4a_1'a_4 + 3\lambda a_3 = 0, \quad a_1a_3' + 2a_2a_4' + \tag{20}$$
$$3a_1'a_3 + 4a_2'a_4' + 2\lambda a_2 = 0$$
$$a_1'a_2' + 2a_2a_3' + 3a_3a_4' + 2a_1'a_2' + 3a_2'a_3 + 4a_3'a_4 + \lambda a_1 = 0$$
$$a_1a_1' + 2a_2a_2' + 3a_3a_3' + 4a_4a_4' = \frac{\mu}{\lg \delta - \lg a_1}$$
$$t = 0,\ a_1(t) = a_1^*, \quad a_2(t) = a_2^*, \quad a_3(t) = a_3^*,\ a_4(t) = a_4^*$$

Note that some of the coefficients may be zero. If Eqs. (20) are satisfied, we have the following equality

$$\operatorname{Re}\left[\frac{d\bar{z}}{\partial t}\zeta\frac{\partial z}{\partial \zeta} - i\lambda\zeta\frac{\partial z}{\partial \zeta}\right]_{\zeta=e^{i\vartheta}} \frac{\mu}{\lg \delta - \lg a_1} = 4a_4\cos 4\nu \tag{21}$$

If the system of equations is restricted to n functions, then generally

$$\operatorname{Re}\left[\frac{\partial \bar{z}}{\partial t}\zeta\frac{\partial z}{\partial \zeta} - i\lambda\zeta\frac{\partial z}{\partial \zeta}\right]_{\zeta=e^{i\vartheta}} - \frac{\mu}{\lg \delta - \lg a_1} = na_n\cos n\nu \tag{22}$$

If the region boundary is smooth, a_n represents a coefficient of a Fourier expansion of some function whose derivative exists everywhere, and the following estimate is valid:

$$|a_n| < c / n^2$$

Consequently the right-hand part of Eq. (22) satisfies the inequality

$$|na_n \cos nv| < c / n \tag{23}$$

with increasing number of equations. Thus this quantity tends with increasing n to zero and the process of successive approximations is convergent.

System (20) can be readily transformed so that the first derivatives of unknown functions are expressed in terms of the functions themselves. In that case

$$a_1' = \frac{\Delta_1}{\Delta}, \quad a_2' = \frac{\Delta_2}{\Delta}, \quad \ldots, \quad a_n' = \frac{\Delta_n}{\Delta} \tag{24}$$

$$t = 0, \quad a_1 = a_1^*, \quad a_2 = a^*, \ldots, \quad a_n = a_n^*$$

where Δ is the determinant obtained from the system of equations and $\Delta_1, \Delta_2, \ldots, \Delta_n$ are the related subdeterminants.

The law of construction of the matrix of coefficients for any number of equations is fairly simple. Let us show the method of forming such matrices on the example of four functions

$$\left\| \begin{matrix} 4a_4 & 0 & 0 & a_1 \\ 3a_3 & 4a_4 & a_1 & 2a_2 \\ 2a_2 & 3a_3 + a_1 & 4a_4 + 2a_2 & 3a_3 \\ a_1 & 2a_2 & 3a_3 & 4a_4 \end{matrix} \right\| \tag{25}$$

Along the diagonals the matrix elements are the same and are added at intersections of diagonals.

Note that in the considered case of filtration of heavy, as well as of weightless fluid the boundary of the region occupied by the fluid (in case of the model used in the classical filtration theory where inertia is disregarded) cannot reach the drain, since the region becomes multivalent.

If the contour reaches the drain, to which the ζ-plane corresponds the coordinate origin, we have

$$|z(\zeta, t)|_{\min} = 0$$

However according to the Koebe theory for one-sheeted functions

$$|z(\zeta, t)|_{\min} > {}^1/_4 \, |a_1(t)|$$

The right-hand part of this inequality is positive and nonzero, which shows that at some instant the univalence is lost. For a weightless fluid this phenomenon was analyzed in [2]. Related computations and successive contour shapes are given in [4].

The solution of the nonlinear problem of heavy fluid filtration, derived on the basis of the classical filtration theory is fairly satisfactory up to a certain instant of time. This must be borne in mind when attempting to obtain a numerical or approximate solution of this problem.

REFERENCES

1. Galin, L.A., Certain problems of unstable motion of ground waters. PMM Vol. 15, №4, 1951.

2. Galin, L.A., Unstable filtration with a free surface. Dokl. Akad. Nauk SSSR,

Vol.47, №4, 1945.
3. Vinogradov, Iu. P., On a certain problem of filtration of heavy fluid with a free surface. Uch. Zap. Tomsk. Univ., № 17, 1952.
4. Kotelenets, N. A., On the computation of unsteady filtration at constant pressure at the boundary. Inzh. Sb., Vol. 12, 1952.

Translated by J. J. D.

UDC 532.593

ON STABLE COMPOSITE CAPILLARY-GRAVITATIONAL WAVES OF FINITE AMPLITUDE ON THE SURFACE OF A FLUID OF FINITE DEPTH

PMM Vol. 39, № 6, 1975, pp. 1023-1031
Ia. I. SEKERZH-ZEN'KOVICH
(Moscow)
(Received June 4, 1975)

The problem of stable plane capillary-gravitational waves of finite amplitude on the surface of a perfect incompressible fluid stream of finite depth is considered. It is assumed that the waves are induced by pressure periodically distributed along the free surface, and that these, unlike induced waves, do not vanish when the pressure becomes constant, are transformed into free waves. Such waves are called composite; they exist similarly to free waves, for particular values of velocity of the stream.

The problem, which is rigorously stated, reduces to solving a system of four nonlinear equations for two functions and two constants. One of the equations is integral and the remaining are transcendental. Pressure on the surface is defined by an infinite trigonometric series whose coefficients are proportional to integral powers of some dimensionless small parameter; these powers are by two units greater than the numbers of coefficients.

The theorem of existence and uniqueness of solution is established, and the method of its proof is indicated. The derivation of solution in any approximation is presented in the form of series in powers of the indicated small parameter. Computation of the first three approximations is carried out to the end, and an approximate equation of the wave profile is presented.

Composite capillary-gravitational waves in the case of fluid of infinite depth were considered by the author in [1].

1. Statement of problem and derivation of basic equations. Let us consider a steady plane-parallel motion of a perfect incompressible heavy fluid of finite and constant depth h bounded from above by a free surface subjected to pressure $p_0 = p_0' + p_0(x)$, where $p_0' = \text{const}$ and $p_0(x)$ is a specified function of the horizontal coordinate x. We assume that the mean velocity c of the stream at the horizontal bottom is constant, is specified and directed from left to right. The term $p_0(x)$ indicates the presence of induced waves at any velocity c. In the absence of $p_0(x)$ free waves appear in the stream at certain particular values of c. Here it is assumed that pressure at the free surface is defined by the two terms. In this case the free surface in coordinates attached to the progressing wave moving at velocity c has the